Name:___________________________

SM3 3.1: Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that polynomials of at least first degree have at least one complex root. This means that we can take a polynomial of degree n and rewrite it as a product of a linear factor, which has form $(ax - b)$, and a polynomial of degree $n - 1$. That process can be repeated until the polynomial is rewritten as the product of linear factors.

$$f(x) = x^n + x^{n-1} + x^{n-2} + \cdots$$

$$f(x) = (a_1x + b_1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots)$$

$$f(x) = (a_1x + b_1)(a_2x + b_2)(x^{n-2} + x^{n-3} + x^{n-4} + \cdots)$$

$$\vdots$$

$$f(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3)\cdots(a_nx + b_n)$$

Example: Find the complete linear factorization for $x^3 - 3x^2 + 7x - 21$

$(x^3 - 3x^2) + (7x - 21)$	Because 1 to -3 has the same ratio as 7 to -21, this is an ideal grouping problem.
$x^2(x - 3) + 7(x - 3)$	Factor the GCF from each group.
$(x - 3)(x^2 + 7)$	Rewrite as product of linear factors and the rest of the polynomial.

Are all of the factors linear? Not yet, we'll need to further factor.

$(x - 3)(x + i\sqrt{7})(x - i\sqrt{7})$	Complex factoring

Example: Find the complete linear factorization for $-5x^4 - 15x^2 - 10$

$-5(x^4 + 3x^2 + 2)$	Factor the GCF from the polynomial
$-5(x^2 + 2)(x^2 + 1)$	Quadratic techniques
$-5(x + i\sqrt{2})(x - i\sqrt{2})(x^2 + 1)$	Complex factoring
$-5(x + i\sqrt{2})(x - i\sqrt{2})(x + i)(x - i)$	Complex factoring

Example: Find the complete linear factorization for $64x^3 - 27$

$(4x - 3)(16x^2 + 12x + 9)$	Sum/Difference of cubes $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
$16(4x - 3)\left(x + \frac{3}{8} - \frac{3i\sqrt{3}}{8}\right)\left(x + \frac{3}{8} + \frac{3i\sqrt{3}}{8}\right)$	Quadratic formula $16x^2 + 12x + 9$ has roots $x = -\frac{3}{8} \pm \frac{3i\sqrt{3}}{8}$ *Don't forget to put the leading coefficient out in front of the complex factoring.

Sometimes, you'll know one or more of the roots or factors before you begin, and can use that to your advantage by performing synthetic division using the known root to determine the factorization.

Example: Find the complete linear factorization for each polynomial with the given factor(s).

$x^3 - 39x - 70; (x + 5)$

$$\begin{array}{c|cccc} \boxed{-5} & 1 & 0 & -39 & -70 \\ & \downarrow & -5 & 25 & 70 \\ \hline & 1 & -5 & -14 & \boxed{0} \end{array}$$

Synthetic division with $x = -5$

$(x + 5)(x^2 - 5x - 14)$

Because $\frac{x^3-39x-70}{x+5} = x^2 - 5x - 14$, we know that $(x + 5)$ and $(x^2 - 5x - 14)$ are the factors of $x^3 - 39x - 70$.

$(x + 5)(x - 7)(x + 2)$

Factor the trinomial

State the total number of roots and possible number of real and imaginary roots for each polynomial.

1) $x^6 + 7x^5 + 4x^3 + x - 1$

2) $x^5 + 4x^4 + 3x^3 - x^2 + 1$

3) $x^3 + 2x - 63$

4) $8x^4 - 64$

Find the complete linear factorization for each polynomial with the given factor(s).

5) $x^3 + 4x^2 + x - 6; (x + 2)$

6) $x^3 - 7x^2 + 2x + 40; (x - 5)$

7) $6x^3 - 29x^2 + 23x + 30; (x - 3)$

8) $3x^3 - 6x^2 - 255x - 462; (x - 11)$

9) $x^4 + 27x^2 - 324; (x + 3)(x - 3)$

10) $x^4 + 10x^3 + 29x^2 + 50x + 120; (x + 4)(x + 6)$

Factor completely

11) $x^4 - 16$

12) $x^3 - 27$

13) $16x^4 - 256$

14) $3x^3 + 24$